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# A simple nonparametric estimator of the bivariate survival function under univariate censoring

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SUMMARY

In the presence of univariate censoring, the bivariate survival function of paired failure times can be expressed as the ratio of the bivariate at-risk probability to the survival function of the censoring time. The use of this natural representation yields a very simple nonparametric estimator for the bivariate survival curve. The estimator is strongly consistent and, upon proper normalization, converges weakly to a zero-mean Gaussian process with an easily estimated covariance function. Numerical studies demonstrate that both the survival curve estimator and its covariance function estimator perform markedly well for practical sample sizes. Applications to the correlation problem and to the interval estimation of the difference in median survival times are also studied.

*Some key words*: Correlated failure times; Matched pairs; Median survival time; Multivariate distribution; Paired failure times; Product-limit estimator; Survival data.

## 1. INTRODUCTION

Bivariate failure time data arise frequently in scientific research because each study subject may experience two types of events or because there exists natural or artificial pairing such that failure times within the same pair are correlated. For example, an HIV-infected person will develop AIDS and eventually expire, and the times to both events are important response variables in AIDS research. Other biomedical examples include times to severe visual loss on the left and right eyes, and times of cancer detection in the left and right breasts. In such studies, either or both failure times may not be observed due to right-censorship.

Nonparametric estimation of the bivariate survival function in the presence of censoring is of great importance in applications. The bivariate estimator is not only useful in predicting the joint survival experience, but also plays crucial roles in estimating the degree of dependence, in model building and testing, and in strengthening marginal analyses. Gill (1992) gave an insightful discussion on why the analysis of censored multivariate data is a 'notoriously tough' problem.

Several estimators for the bivariate survival curve have been suggested, including those of A. Muñoz, given in Stanford University Technical Report 60, Campbell (1981), Campbell & Földes (1982), Langberg & Shaked (1982), Hanley & Parnes (1983), Tsai, Leurgans

& Crowley (1986) and Dabrowska (1988). The Dabrowska estimator, which overcomes some undesirable features, such as nonuniqueness, inconsistency and lack of weak convergence theory, of its predecessors, has won the highest praise among the existing estimators, though two recent estimators due to Prentice & Cai (1992) and Pruitt (1993) appear to be in good contention. Unfortunately, these three estimators and virtually all others are quite complicated and are difficult for practitioners to grasp. Furthermore, the covariance functions of the estimators cannot be estimated analytically.

In many bivariate survival studies, the two failure times are subject to independent censorship by a single censoring variable. This censoring mechanism has been known as univariate censoring, and is typical of the special examples cited in the opening paragraph. Under univariate censoring, there exists a natural representation of the bivariate survival function in terms of the bivariate at-risk probability and the survival function of the censoring time. By making use of this representation, we develop a very simple nonparametric estimator for the bivariate survival curve. The resulting estimator is strongly consistent and, upon proper normalization, converges weakly to a zero-mean Gaussian process with an easily estimated covariance function. Simulation studies demonstrate that both the survival curve estimator and the covariance function estimator perform markedly well for practical sample sizes. Interesting applications of the proposed estimators include the correlation problem and the interval estimation of the difference in median survival times.

## 2. BIVARIATE SURVIVAL FUNCTION ESTIMATOR AND RELATED INFERENCE PROCEDURES

Let $(X_i, Y_i)$ $(i = 1, \ldots, n)$ be $n$ independent and identically distributed pairs of failure times with survival function $F(x, y) = \mathrm{pr}(X \geqslant x, Y \geqslant y)$, and let $C_i$ $(i = 1, \ldots, n)$ be $n$ independent and identically distributed censoring times with survival function $G(t) = \mathrm{pr}(C \geqslant t)$. It is assumed that $C_i$ $(i = 1, \ldots, n)$ are independent of $(X_i, Y_i)$ $(i = 1, \ldots, n)$. The data consist of random vectors $(\tilde{X}_i, \tilde{Y}_i, \delta_i^x, \delta_i^y)$ $(i = 1, \ldots, n)$, where $\tilde{X}_i = X_i \wedge C_i$, $\tilde{Y}_i = Y_i \wedge C_i$, $\delta_i^x = I(X_i \leqslant C_i)$ and $\delta_i^y = I(Y_i \leqslant C_i)$. Here and in the sequel, $I(\,.\,)$ denotes the indicator function, $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

Our estimation of $F(\,.\,,\,.\,)$ will be based on the following representation:

$$F(x, y) = \mathrm{pr}(\tilde{X} \geqslant x, \tilde{Y} \geqslant y)/G(x \vee y),$$

which is a simple consequence of the independence between $(X, Y)$ and $C$. Since $\tilde{X}$ and $\tilde{Y}$ are the observation times, it is natural to estimate $\mathrm{pr}(\tilde{X} \geqslant x, \tilde{Y} \geqslant y)$ by the empirical survival function

$$n^{-1} \sum_{i=1}^{n} I(\tilde{X}_i \geqslant x, \tilde{Y}_i \geqslant y). \tag{2·1}$$

In addition, we may estimate $G(\,.\,)$ from the data $(\tilde{C}_i, \delta_i^c)$ $(i = 1, \ldots, n)$ by the product-limit method, where

$$\tilde{C}_i = C_i \wedge (X_i \vee Y_i) = \tilde{X}_i \vee \tilde{Y}_i, \quad \delta_i^c = I\{C_i \leqslant (X_i \vee Y_i)\} = 1 - \delta_i^x \delta_i^y.$$

To be specific, let $\{c_k; k \geqslant 1\}$ be the ordered sequence of distinct time points where censoring occurs, and let $d_k$ be the number of censored observations at $c_k$. Also let

$$n_k = \sum_{i=1}^{n} I(\tilde{C}_i \geqslant c_k).$$

Then the product-limit estimator for $G(\,.\,)$ is

$$\hat{G}(t) = \prod_{k:\,c_k<t} \frac{n_k - d_k}{n_k}. \tag{2·2}$$

Combining estimators (2·1) and (2·2), we obtain the following estimator for $F(x,y)$:

$$\hat{F}(x,y) = n^{-1} \sum_{i=1}^{n} I(\tilde{X}_i \geqslant x, \tilde{Y}_i \geqslant y)/\hat{G}(x \vee y). \tag{2·3}$$

This bivariate estimator is far simpler than any existing ones and reduces to the usual empirical survival function in the absence of censoring.

Estimator (2·3) is a natural generalization of the univariate product-limit estimator. In the univariate case, the product-limit estimator for the failure time equals the empirical survival function for the observation time divided by the product-limit estimator for the censoring time (Shorack & Wellner, 1986, p.295). Thus, $\hat{F}(t,t)$ becomes the product-limit estimator for $F(t,t)$ if $X \equiv Y$. In addition, $\hat{F}(t,0) + \hat{F}(0,t) - \hat{F}(t,t)$ is the product-limit estimator for $\mathrm{pr}\,(X \vee Y \geqslant t)$, or $F(t,0) + F(0,t) - F(t,t)$, calculated from $\{\tilde{X}_i \vee \tilde{Y}_i, \delta_i^x \delta_i^y\}$ $(i = 1, \ldots, n)$. Because of its intimate relationship with the univariate product-limit estimator, we expect the bivariate estimator (2·3) to have good efficiency, at least when $X$ and $Y$ are highly correlated or when censoring is light.

We show in the Appendix that the process $n^{\frac{1}{2}}\{\hat{F}(\,.\,,.\,) - F(\,.\,,.\,)\}$ converges weakly to a zero-mean Gaussian process, the asymptotic covariance between $n^{\frac{1}{2}}\{\hat{F}(x_1,y_1) - F(x_1,y_1)\}$ and $n^{\frac{1}{2}}\{\hat{F}(x_2,y_2) - F(x_2,y_2)\}$ being

$$V(x_1,y_1;x_2,y_2) = \frac{F(x_1 \vee x_2, y_1 \vee y_2)}{G\{(x_1 \vee y_1) \wedge (x_2 \vee y_2)\}} - F(x_1,y_1)F(x_2,y_2)\left[1 - \int_0^{(x_1 \vee y_1)\wedge(x_2 \vee y_2)} \frac{dG(u)}{G^2(u)\,\mathrm{pr}\,\{(X \vee Y) \geqslant u\}}\right]. \tag{2·4}$$

To be precise, the weak convergence is restricted to $(x,y) \in [0,\tau]^2$, where $\tau$ satisfies $F(\tau,\tau)G(\tau) > 0$. Since the integral in (2·4) is always nonpositive, we observe that the variance

$$V(x,y) \leqslant \frac{F(x,y)}{G(x \vee y)} - F^2(x,y). \tag{2·5}$$

Interestingly, the right-hand side of (2·5) is the asymptotic variance of the estimator

$$\tilde{F}(x,y) = n^{-1} \sum_{i=1}^{n} I(\tilde{X}_i \geqslant x, \tilde{Y}_i \geqslant y)/G(x \vee y),$$

which one might be tempted to use if $G(\,.\,)$ is known. As indicated in the Appendix, the covariance function (2·4) can be consistently estimated by

$$\hat{V}(x_1,y_1;x_2,y_2) = \frac{\hat{F}(x_1 \vee x_2, y_1 \vee y_2)}{\hat{G}\{(x_1 \vee y_1) \wedge (x_2 \vee y_2)\}} - \hat{F}(x_1,y_1)\hat{F}(x_2,y_2)\left[1 - \sum_{k:\,c_k<(x_1 \vee y_1)\wedge(x_2 \vee y_2)} \frac{n\{\hat{G}(c_k+) - \hat{G}(c_k)\}}{\hat{G}(c_k)\sum_{i=1}^{n} I(\tilde{X}_i \vee \tilde{Y}_i \geqslant c_k)}\right]. \tag{2·6}$$

If $G(\,.\,)$ is discontinuous, then $G^2(u)$ in (2·4) should be replaced by $G(u)G(u+)$ and $\hat{G}(c_k)$ in the denominator of the last term in (2·6) by $\hat{G}(c_k+)$.

The bivariate survival function estimator is valuable not only in predicting the joint survival experience, but also in subsequent statistical analyses. For example, it is natural to estimate the correlation between $X$ and $Y$ by the empirical correlation of the estimated survival function (2·3). In particular, $E(XY)$ is estimated by

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{X}_i \tilde{Y}_j \{\hat{F}(\tilde{X}_i, \tilde{Y}_j) + \hat{F}(\tilde{X}_i+, \tilde{Y}_j+) - \hat{F}(\tilde{X}_i+, \tilde{Y}_j) - \hat{F}(\tilde{X}_i, \tilde{Y}_j+)\}.$$

It should be noted that the uniform consistency of $\hat{F}(\,.\,,\,.\,)$ over the entire support of $F(\,.\,,\,.\,)$ is essential to the consistency of the correlation estimator. This requirement will be met if the support of $C$ properly contains those of $X$ and $Y$, in which case $\hat{G}(\,.\,)$ has been known to be a consistent estimator of $G(\,.\,)$ up to the last observation time (Wang, 1987).

Recently, Su & Wei (1993) proposed a simple and purely nonparametric approach for comparing the medians of two independent failure time variables. We now extend their results to the case of dependent data by making use of formulae (2·3) and (2·6). Let $\xi_x$ and $\xi_y$ be the medians of $X$ and $Y$, and let $\eta = \xi_x - \xi_y$. We estimate $\eta$ by $\hat{\eta} = \hat{\xi}_x - \hat{\xi}_y$, where

$$\hat{\xi}_x = \inf\{t : \hat{F}(t, 0) \leqslant \tfrac{1}{2}\}, \quad \hat{\xi}_y = \inf\{t : \hat{F}(0, t) \leqslant \tfrac{1}{2}\}.$$

The random variable $n^{\frac{1}{2}}(\hat{\eta} - \eta)$ can be shown to be asymptotically zero-mean normal by simple probability arguments (Reid, 1981). The limiting variance, however, involves the unknown density function for $F$, which cannot be reliably estimated. Thus it is difficult to make inference about $\eta$ by using the asymptotic distribution of $\hat{\eta}$ directly. We will instead construct the inference procedures from the distributional properties of $\hat{F}(\,.\,,\,.\,)$.

Define the quadratic form

$$Q(\eta;\xi) = \begin{pmatrix} \hat{F}(\xi, 0) - \frac{1}{2} \\ \hat{F}(0, \xi - \eta) - \frac{1}{2} \end{pmatrix}' \begin{pmatrix} \hat{V}(\hat{\xi}_x, 0) & \hat{V}(\hat{\xi}_x, 0; 0, \hat{\xi}_y) \\ \hat{V}(\hat{\xi}_x, 0; 0, \hat{\xi}_y) & \hat{V}(0, \hat{\xi}_y) \end{pmatrix}^{-1} \begin{pmatrix} \hat{F}(\xi, 0) - \frac{1}{2} \\ \hat{F}(0, \xi - \eta) - \frac{1}{2} \end{pmatrix}.$$

For making inference about $\eta$, the parameter $\xi$ in $Q(\eta;\xi)$ is a nuisance. A natural way of eliminating this nuisance parameter is to minimize $Q(\eta;\xi)$ with respect to $\xi$. The resulting quantity, denoted by $W(\eta)$, is the so-called minimum-dispersion statistic (Basawa & Koul, 1988). By the arguments given in the Appendix of Su & Wei (1993) and in Appendix 2 of Wei, Ying & Lin (1990), the test statistic $W(\eta_0)$ is asymptotically chi-squared on 1 degree of freedom under the hypothesis $H_0 : \eta = \eta_0$. A confidence interval for $\eta$ can then be obtained by inverting the $W$ test. Specifically, the $(1-\alpha)100\%$ confidence interval for $\eta$ is $\{\eta : w(\eta) < \chi_1^2(1-\alpha)\}$, where $w(\eta)$ is the observed value of $W(\eta)$ and $\chi_1^2(1-\alpha)$ is the $(1-\alpha)100$th percentile of the chi-squared distribution on 1 degree of freedom. The foregoing procedures are asymptotically valid even if the two marginal distributions differ in shape.

## 3. Numerical studies

### 3·1. *A real example*

As an illustration, let us consider the well-known matched pairs data of Holt & Prentice (1974), which consist of survival times, in days, of closely and poorly matched skin grafts on the same burn patient. With the minor modifications made by Woolson & Lachenbruch (1980) these data are reproduced in Table 1. There were only 11 patients, the survival times

Table 1. *Days of survival of skin grafts on burn patients*

| | Patient, $i$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| Survival of close match, $X_i$ | 37 | 19 | $57^+$ | 93 | 16 | 22 | 20 | 18 | 63 | 29 | $60^+$ |
| Survival of poor match, $Y_i$ | 29 | 13 | 15 | 26 | 11 | 17 | 26 | 21 | 43 | 15 | 40 |

$^+$ Indicates censored.

Table 2. *Survival function estimates and their estimated variances for skin grafts*

| $x$ | $y = 11$ | $y = 13$ | $y = 15$ | $y = 17$ | $y = 21$ | $y = 26$ | $y = 29$ | $y = 40$ | $y = 43$ |
|---|---|---|---|---|---|---|---|---|---|
| 16 | 1·00 | 0·909 | 0·818 | 0·636 | 0·545 | 0·455 | 0·273 | 0·182 | 0·091 |
| | 0·000 | 0·008 | 0·014 | 0·021 | 0·023 | 0·023 | 0·018 | 0·014 | 0·008 |
| 18 | 0·909 | 0·909 | 0·818 | 0·636 | 0·545 | 0·455 | 0·273 | 0·182 | 0·091 |
| | 0·008 | 0·008 | 0·014 | 0·021 | 0·023 | 0·023 | 0·018 | 0·014 | 0·008 |
| 19 | 0·818 | 0·818 | 0·727 | 0·545 | 0·455 | 0·455 | 0·273 | 0·182 | 0·091 |
| | 0·014 | 0·014 | 0·018 | 0·023 | 0·023 | 0·023 | 0·018 | 0·014 | 0·008 |
| 20 | 0·727 | 0·727 | 0·727 | 0·545 | 0·455 | 0·455 | 0·273 | 0·182 | 0·091 |
| | 0·018 | 0·018 | 0·018 | 0·023 | 0·023 | 0·023 | 0·018 | 0·014 | 0·008 |
| 22 | 0·636 | 0·636 | 0·636 | 0·455 | 0·364 | 0·364 | 0·273 | 0·182 | 0·091 |
| | 0·021 | 0·021 | 0·021 | 0·023 | 0·021 | 0·021 | 0·018 | 0·014 | 0·008 |
| 29 | 0·545 | 0·545 | 0·545 | 0·364 | 0·364 | 0·364 | 0·273 | 0·182 | 0·091 |
| | 0·023 | 0·023 | 0·023 | 0·021 | 0·021 | 0·021 | 0·018 | 0·014 | 0·008 |
| 37 | 0·455 | 0·455 | 0·455 | 0·364 | 0·364 | 0·364 | 0·273 | 0·182 | 0·091 |
| | 0·023 | 0·023 | 0·023 | 0·021 | 0·021 | 0·021 | 0·018 | 0·014 | 0·008 |
| 57 | 0·364 | 0·364 | 0·364 | 0·273 | 0·273 | 0·273 | 0·182 | 0·182 | 0·091 |
| | 0·021 | 0·021 | 0·021 | 0·018 | 0·018 | 0·018 | 0·014 | 0·014 | 0·008 |
| 60 | 0·364 | 0·364 | 0·364 | 0·364 | 0·364 | 0·364 | 0·242 | 0·242 | 0·121 |
| | 0·024 | 0·024 | 0·024 | 0·024 | 0·024 | 0·024 | 0·020 | 0·020 | 0·012 |
| 63 | 0·364 | 0·364 | 0·364 | 0·364 | 0·364 | 0·364 | 0·182 | 0·182 | 0·182 |
| | 0·031 | 0·031 | 0·031 | 0·031 | 0·031 | 0·031 | 0·024 | 0·024 | 0·024 |
| 93 | 0·182 | 0·182 | 0·182 | 0·182 | 0·182 | 0·182 | 0·000 | 0·000 | 0·000 |
| | 0·024 | 0·024 | 0·024 | 0·024 | 0·024 | 0·024 | 0·000 | 0·000 | 0·000 |

of two closely matched grafts being censored. This small data set has been chosen because it enables us to see clearly how the proposed methods work, though one would not expect the asymptotic approximations to be accurate for such small sample sizes.

Table 2 displays the estimates of the bivariate survival functions and their estimated variances at all observed failure time points for the skin graft data. Thus the probability that both grafts survive beyond 20 days is estimated at 0·455 with a standard error estimate of 0·15. The correlation between the survival times of the closely and poorly matched grafts is estimated at 0·52. The median survival times are estimated at 29 and 21 for the close and poor matches, respectively, and the covariance estimate between $\hat{F}(29, 0)$ and $\hat{F}(0, 21)$ is about 0·014. Then the 95% confidence interval for the difference of the two medians is found to be $(-6, 46)$.

### 3·2. *Simulation studies*

Two sets of 1000 simulations were carried out to assess the performance of the survival function estimator (2·3) and the covariance function estimator (2·6) in samples of moderate size. Each simulation sample consisted of 60 pairs of failure times with unit exponential

Table 3. *Simulation summary statistics for the bivariate survival function estimator under an independent bivariate exponential model and under a Gumbel bivariate exponential model: (a) theoretical survival probabilities, (b) empirical means of estimated survival probabilities, (c) empirical variances of estimated survival probabilities*, and (d) empirical means of variance estimates for estimated survival probabilities**

| | | Independence model | | | | Gumbel model | | | |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | $y$ | 0 | 0·2231 | 0·5108 | 0·9163 | 0 | 0·2231 | 0·5108 | 0·9163 |
| 0 | (*a*) | 1 | 0·8 | 0·6 | 0·4 | 1 | 0·8 | 0·6 | 0·4 |
| | (*b*) | 1 | 0·803 | 0·605 | 0·406 | 1 | 0·798 | 0·594 | 0·396 |
| | (*c*) | 0 | 0·281 | 0·490 | 0·602 | 0 | 0·289 | 0·479 | 0·575 |
| | (*d*) | 0 | 0·291 | 0·499 | 0·598 | 0 | 0·295 | 0·497 | 0·583 |
| 0·2231 | (*a*) | 0·8 | 0·64 | 0·48 | 0·32 | 0·8 | 0·666 | 0·518 | 0·358 |
| | (*b*) | 0·799 | 0·642 | 0·482 | 0·324 | 0·799 | 0·664 | 0·514 | 0·355 |
| | (*c*) | 0·301 | 0·424 | 0·530 | 0·550 | 0·300 | 0·419 | 0·496 | 0·555 |
| | (*d*) | 0·296 | 0·422 | 0·524 | 0·551 | 0·294 | 0·409 | 0·519 | 0·564 |
| 0·5108 | (*a*) | 0·6 | 0·48 | 0·36 | 0·24 | 0·6 | 0·518 | 0·418 | 0·298 |
| | (*b*) | 0·596 | 0·479 | 0·360 | 0·241 | 0·596 | 0·515 | 0·412 | 0·294 |
| | (*c*) | 0·540 | 0·543 | 0·512 | 0·497 | 0·545 | 0·557 | 0·505 | 0·511 |
| | (*d*) | 0·502 | 0·524 | 0·485 | 0·464 | 0·495 | 0·518 | 0·507 | 0·518 |
| 0·9163 | (*a*) | 0·4 | 0·32 | 0·24 | 0·16 | 0·4 | 0·358 | 0·298 | 0·218 |
| | (*b*) | 0·399 | 0·321 | 0·241 | 0·162 | 0·400 | 0·357 | 0·294 | 0·215 |
| | (*c*) | 0·605 | 0·565 | 0·505 | 0·383 | 0·604 | 0·579 | 0·547 | 0·444 |
| | (*d*) | 0·595 | 0·548 | 0·464 | 0·348 | 0·583 | 0·565 | 0·517 | 0·428 |

* Rows (*c*) and (*d*) have been multiplied by 100

marginal distributions. These values were subject to censorship by means of an independent exponentially distributed censoring time with hazard rate of 0·5. Hence each failure time has a $\frac{1}{3}$ marginal probability of being censored. In the two simulation studies, the pairs of failure times were, respectively, independent and distributed according to the Gumbel (1960) bivariate exponential with correlation 0·25.

Table 3 summarizes the simulation results. The estimates are given at pairs of time points where $x$ and $y$ take values 0, 0·2231, 0·5108 or 0·9163, corresponding to marginal survival probabilities of 1, 0·8, 0·6 and 0·4. The bias of the survival function estimator is negligible. The variance estimator appears to be more biased, but none of the estimated biases are greater than 10%.

Prentice & Cai (1992) conducted three simulation studies to examine the properties of their estimator and the Dabrowska estimator. The sampling schemes for their first two studies were identical to ours except that they generated pairs of independent exponential censoring times. Comparisons of our findings with theirs suggest that the new estimator behaves similarly to the Prentice–Cai and Dabrowska estimators.

In our two simulation studies, we also evaluated the correlation estimator and the interval estimator for the median difference described in the last section. The correlation estimator was found to be nearly unbiased under the independence model. For the Gumbel model, however, the empirical mean of the correlation estimator was about 0·285. Additional experiments indicated that the bias of the estimator is reduced rapidly as the amount of censoring decreases and is reduced steadily as the sample size increases. It should be pointed out that the biasedness of the correlation estimator in the presence of moderate to heavy censoring is unavoidable due to the difficulty in estimating the right-tail of the failure time distribution.

For comparing median survival times, our simulation results were similar to those of Su & Wei (1993). The inference procedures are rather conservative unless the effective sample size is large. The conservativeness is attributed to the fact that the normal approximation to the distribution of the median estimator tends to be poor because of the discrete nature of the survival function estimator.

## 4. Remarks

The proposed bivariate estimator is less general than several existing ones due to the restrictive assumption on censoring. Its great simplicity and desirable asymptotic and small-sample properties, however, indicate that it will be very useful in the case of univariate censoring. When the two censoring time variables $C^x$ and $C^y$ are, with high probability, almost equal, one may set $C = C^x \wedge C^y$ at the expense of slight loss in efficiency. A FORTRAN program that implements the proposed procedures is available from the first author.

In some applications, there are more than two components of failure times. It is straightforward to generalize the results presented in § 2 to higher dimensions.

We have seen the important roles played by the bivariate survival function estimator in predicting the survival experience and in estimating the correlation and the difference of median survival times. We are currently investigating the use of the proposed estimator in constructing optimal two-sample nonparametric tests for censored paired data (O'Brien & Fleming, 1987) and in improving the efficiency of the marginal regression analysis with multivariate survival data (Wei, Lin & Weissfeld, 1989).

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## Appendix

### *Proof of the asymptotic theory*

Since it is the bivariate empirical survival function, (2·1) is strongly consistent and

$$n^{-\frac{1}{2}}\sum\{I(\tilde{X}_i \geqslant x, \tilde{Y}_i \geqslant y) - \text{pr}\,(\tilde{X} \geqslant x, \tilde{Y} \geqslant y)\}$$

converges weakly to a zero-mean Gaussian process (Shorack & Wellner, 1986, Ch.26). The product-limit estimator $\hat{G}(\,.\,)$ is also strongly consistent and $n^{\frac{1}{2}}\{\hat{G}(\,.\,) - G(\,.\,)\}$ converges weakly to a zero-mean Gaussian process (Breslow & Crowley, 1974). From these well-known facts follow the strong consistency of the bivariate survival function estimator $\hat{F}(\,.\,,.\,)$ and the weak convergence of $n^{\frac{1}{2}}\{\hat{F}(\,.\,,.\,) - F(\,.\,,.\,)\}$ to a zero-mean Gaussian process.

To elaborate on the weak convergence of $n^{\frac{1}{2}}\{\hat{F}(\,.\,,.\,) - F(\,.\,,.\,)\}$, we observe that

$$n^{\frac{1}{2}}\{\hat{F}(x,y) - F(x,y)\} = \frac{1}{\hat{G}(x \vee y)} n^{-\frac{1}{2}} \sum_{i=1}^{n}\{I(\tilde{X}_i \geqslant x, \tilde{Y}_i \geqslant y) - \text{pr}\,(\tilde{X} \geqslant x, \tilde{Y} \geqslant y)\} + \frac{F(x,y)}{\hat{G}(x \vee y)} n^{\frac{1}{2}}\{G(x \vee y) - \hat{G}(x \vee y)\}. \quad (\text{A·1})$$

Due to the properties of the empirical survival function and the product-limit estimator, the right-hand side of (A·1) is asymptotically equivalent to

$$\frac{1}{G(x \vee y)} n^{-\frac{1}{2}} \sum_{i=1}^{n} \{I(\tilde{X}_i \geqslant x, \tilde{Y}_i \geqslant y) - \operatorname{pr}(\tilde{X} \geqslant x, \tilde{Y} \geqslant y)\}$$

$$+ F(x,y) n^{-\frac{1}{2}} \sum_{i=1}^{n} \int_0^{x \vee y} \frac{\delta_i^c \, dI(\tilde{C}_i \leqslant u) - I(\tilde{C}_i \geqslant u)\, d\Lambda_c(u)}{G(u) \operatorname{pr}\{(X \vee Y) \geqslant u\}} = A(x,y) + B(x,y), \qquad (\text{A·2})$$

say, where $\Lambda_c(\,.\,)$ is the cumulative hazard function for the censoring time variable $C$. It is helpful to note that

$$\delta_i^c I(\tilde{C}_i \leqslant t) - \int_0^t I(\tilde{C}_i \geqslant u)\, d\Lambda_c(u) \quad (i = 1, \ldots, n)$$

are martingales.

The independent identically distributed representation given in (A·2) and the tightness of

$$n^{\frac{1}{2}}\{\hat{G}(\,.\,) - G(\,.\,)\}, \quad n^{-\frac{1}{2}} \sum \{I(\tilde{X}_i \geqslant \,.\,, \tilde{Y}_i \geqslant \,.\,) - \operatorname{pr}(\tilde{X} \geqslant \,.\,, \tilde{Y} \geqslant \,.\,)\}$$

imply that $n^{\frac{1}{2}}\{\hat{F}(\,.\,,\,.\,) - F(\,.\,,\,.\,)\}$ converges weakly to a zero-mean Gaussian process. Furthermore, we obtain the following covariance expressions for $A$ and $B$ by using the properties of martingales and some elementary probability arguments:

$$E\{A(x_1,y_1)A(x_2,y_2)\}$$
$$= \frac{G(x_1 \vee y_1 \vee x_2 \vee y_2) F(x_1 \vee x_2, y_1 \vee y_2) - G(x_1 \vee y_1) G(x_2 \vee y_2) F(x_1,y_1) F(x_2,y_2)}{G(x_1 \vee y_1) G(x_2 \vee y_2)}$$
$$= \frac{F(x_1 \vee x_2, y_1 \vee y_2)}{G\{(x_1 \vee y_1) \wedge (x_2 \vee y_2)\}} - F(x_1,y_1)F(x_2,y_2), \qquad (\text{A·3})$$

$$E\{B(x_1,y_1)B(x_2,y_2)\} = F(x_1,y_1)F(x_2,y_2) \int_0^{(x_1 \vee y_1) \wedge (x_2 \vee y_2)} \frac{-dG(u)}{G^2(u) \operatorname{pr}\{(X \vee Y) \geqslant u\}}, \qquad (\text{A·4})$$

$$E\{A(x_1,y_1)B(x_2,y_2)\} = -E\{B(x_1,y_1)B(x_2,y_2)\}. \qquad (\text{A·5})$$

To see why (A·5) is true, note that

$$E\left[\{I(\tilde{X} \geqslant x_1, \tilde{Y} \geqslant y_1) - \operatorname{pr}(\tilde{X} \geqslant x_1, \tilde{Y} \geqslant y_1)\} \int_0^{x_2 \vee y_2} \frac{\delta^c \, dI(\tilde{C} \leqslant u) - I(\tilde{C} \geqslant u)\, d\Lambda_c(u)}{G(u) \operatorname{pr}\{(X \vee Y) \geqslant u\}}\right]$$
$$= E\left[I(\tilde{X} \geqslant x_1, \tilde{Y} \geqslant y_1) \int_0^{(x_1 \vee y_1) \wedge (x_2 \vee y_2)} \frac{\delta^c \, dI(\tilde{C} \leqslant u) - I(\tilde{C} \geqslant u)\, d\Lambda_c(u)}{G(u) \operatorname{pr}\{(X \vee Y) \geqslant u\}}\right]$$
$$= -E\left[I(\tilde{X} \geqslant x_1, \tilde{Y} \geqslant y_1) \int_0^{(x_1 \vee y_1) \wedge (x_2 \vee y_2)} \frac{d\Lambda_c(u)}{G(u) \operatorname{pr}\{(X \vee Y) \geqslant u\}}\right]$$
$$= -F(x_1,y_1) G(x_1 \vee y_1) \int_0^{(x_1 \vee y_1) \wedge (x_2 \vee y_2)} \frac{d\Lambda_c(u)}{G(u) \operatorname{pr}\{(X \vee Y) \geqslant u\}}.$$

Combining expressions (A·3) to (A·5), we get the covariance function (2·4). In addition, the consistency of the covariance function estimator (2·6) follows from the consistency of $\hat{F}(\,.\,,\,.\,)$ and $\hat{G}(\,.\,)$.

For simplicity of presentation, we made an implicit assumption that $G(\,.\,)$ is continuous in the above derivation. If $G(\,.\,)$ has points of discontinuity, then $G^2(u)$ in (2·4) should be replaced by $G(u)G(u+)$ and $\hat{G}(c_k)$ in the denominator of the last term in (2·6) should be replaced by $\hat{G}(c_k+)$. Finally, it can be verified that the asymptotic theory developed here also holds under the fixed-censorship model, but $G(t)$ then should be replaced by the probability limit of $n^{-1} \sum I(C_i \geqslant t)$.

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